Short Papers

Expansions for the Capacitance of a Square in a Square with a Comparison

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Abstract—**Expansions are given for the capacitance per unit length for the geometry in which two infinite square cylinders are placed concentric with each other with their sides parallel to each other. A comparison is made with an expansion for the capacitance of the Bowman squares.**

I. Introduction

About 50 years ago, Bowman [1] determined the capacitance of the geometry in Fig. 1 by means of a conformal transformation which maps the quadrilateral, ABCD, in the z-plane of Fig. 1 onto the rectangle ABCD in the w-plane of Fig. 1. For the case which he considered explicitly [1, (20)]

$$\frac{L(\lambda)}{L'(\lambda)} = \frac{1+\rho}{1-\rho} \tag{1}$$

if $\rho = a/b$. Here L and L' are the complete elliptic integrals of modulus λ. Thus, if r' is defined by

$$r' = \exp(-\pi L/L') \tag{2}$$

to avoid confusion with a q to be used later, from Hancock [2]

$$\sqrt{\lambda} = \frac{1 - 2r' + 2r'^4 - 2r'^9 + \cdots}{1 + 2r' + 2r'^4 + 2r'^9 + \cdots}. \tag{3}$$

The equation relating λ to the modulus k, which determines the complete elliptic integrals $K(k)$ and $K'(k)$ is [1, (5)]. This equation may be solved in two steps

$$k_0 = 2\lambda^2 - 1; \qquad k_0 = \frac{2\sqrt{k}}{1+k}. \tag{4}$$

Notice that $\lambda \geqq \sqrt{.5}$ from (3) and (1) since $r' < .044$. If ϵ is defined by

$$\epsilon = \frac{1}{2}\left(\frac{1-\sqrt{k_0}}{1+\sqrt{k_0}}\right) \tag{5}$$

then, from Whittaker and Watson [3]

$$q' = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + \cdots. \tag{6}$$

Finally, the capacitance of the square coax is given by

$$C_0 = 8\frac{K(k)}{K'(k)} = 4\frac{K(k_0)}{K'(k_0)} = -4\ln(q')/\pi. \tag{7}$$

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If k_0 from (4) is greater that $\sqrt{.5}$, k_0' may be substituted in (5) and (6) to find q. Then

$$C_0 = -16\pi/\ln(q). \tag{8}$$

Values obtained from this equation are given in the middle row of Table I.

II. The Expansion in Powers of ρ

For this purpose, (1)–(4) are expanded in powers of ρ as follows

$$\frac{L(\lambda)}{L'(\lambda)} = 1 + 2\sum_{i=1}^{\infty}\rho^i \tag{9}$$

$$\begin{aligned} r' = {} & .043214 - .27152\rho + .58149\rho^2 - .35204\rho^3 \\ & - .26582\rho^4 + .11995\rho^5 + .25152\rho^6 + .11973\rho^7 \\ & - .073144\rho^8 - .17209\rho^9 - .14312\rho^{10} - .037658\rho^{11} \\ & + .069958\rho^{12} + .12723\rho^{13} + .11925\rho^{14} + \cdots \end{aligned} \tag{10}$$

$$\begin{aligned} \sqrt{\lambda} = {} & .84090 + .92013\rho - 1.51023\rho^2 - .55084\rho^3 \\ & + 2.86302\rho^4 + .97831\rho^5 - 4.38416\rho^6 - 2.94151\rho^7 \\ & + 7.35630\rho^8 + 4.10501\rho^9 - 11.0331\rho^{10} - 5.10871\rho^{11} \\ & + 16.3695\rho^{12} + 8.51710\rho^{13} - 25.0456\rho^{14} + \cdots \end{aligned} \tag{11}$$

$$\begin{aligned} \lambda^2 = {} & .5 + 2.18844\rho - 10.4810\rho^3 + 43.5036\rho^5 - 144.242\rho^7 \\ & + 413.027\rho^9 - 1081.83\rho^{11} + 2648.37\rho^{13} + \cdots \end{aligned}$$

$$\begin{aligned} k_0' = {} & 4.37688\rho - 20.9620\rho^3 + 87.0071\rho^5 - 288.485\rho^7 \\ & + 826.055\rho^9 - 2163.66\rho^{11} + 5296.74\rho^{13} + \cdots \end{aligned} \tag{12}$$

$$\begin{aligned} k_0^2 = {} & 19.1571\rho^2 - 183.497\rho^4 + 1201.05\rho^6 - 6173.02\rho^8 \\ & + 26895.8\rho^{10} - 103772\rho^{12} + 364044\rho^{14} + \cdots \end{aligned} \tag{13}$$

Now

$$\begin{aligned} \frac{K'(k_0)}{K(k_0)} = {} & \ln\left(\frac{16}{k_0^2}\right) - \frac{1}{2}\left[k_0^2 + \frac{13}{32}k_0^4 + \frac{23}{96}k_0^6 + \frac{2701}{16384}k_0^8 \right. \\ & \left. + \frac{5057}{40960}k_0^{10} + \frac{76715}{786432}k_0^{12} + \cdots\right].^1 \end{aligned} \tag{14}$$

Then substituting (5) into (6) and making use of the relationship between k and k_0, it is found that

$$\begin{aligned} C_0 = {} & 8\pi/(4\ln(1/\rho) - .36017 + .76457\rho^4 \\ & - 1.3098\rho^8 + 3.3689\rho^{12} + \cdots). \end{aligned} \tag{15}$$

The values obtained from this equation are tabulated as $C(\rho)$ in the first row of Table I.

[1] [4, (27)]

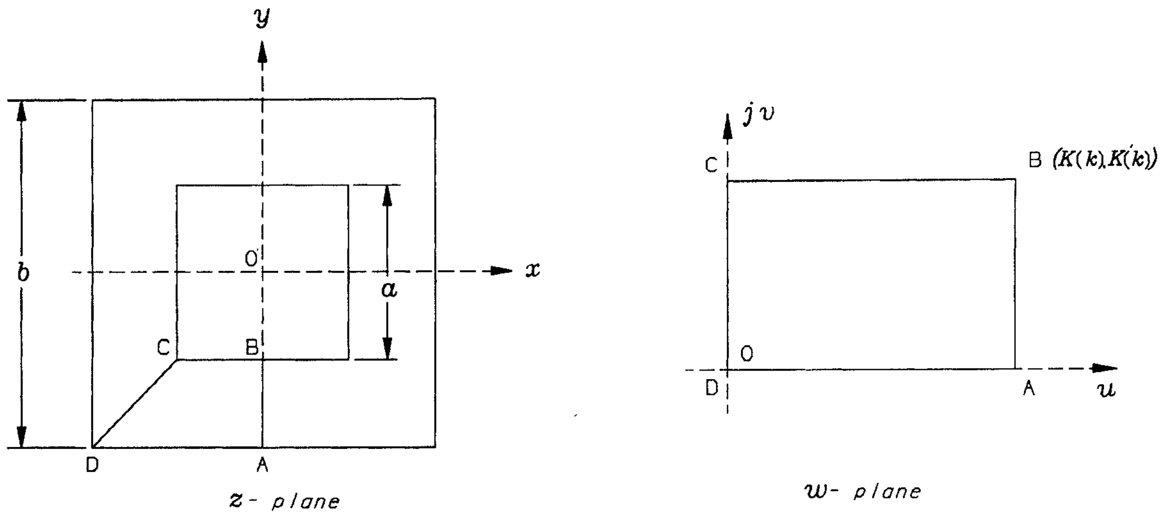

Fig. 1. z and w complex planes.

TABLE I
CAPACITANCE OF SQUARE COAX

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9
$C(\rho)$	2.8398	4.1345	5.6328	7.5616	10.2336	14.2193			
Co	2.8398	4.1345	5.6328	7.5615	10.2341	14.2349	20.9016	34.2349	
$C(\delta)$	2.8470	4.1346	5.6328	7.5615	10.2341	14.2349	20.9016	34.2349	74.2349
δ	.9	.8	.7	.6	.5	.4	.3	.2	.1

III. THE EXPANSION IN TERMS OF δ

If $\delta = 1 - \rho$, then

$$L(\lambda)/L'(\lambda) = 2/\delta - 1 \quad (16)$$

and

$$r' = \exp[-\pi(2/\delta - 1)]. \quad (17)$$

The expansion which results from the substitution of (16) into (3)–(7), may be expressed as a power series in r' with a logarithmic singularity at $r' = 0$. This, because of (16), is equivalent to a pole for $\delta = 0$. The result is a mixed expansion in terms of δ and r'.

We have

$$\sqrt{\lambda} = 1 - 4r' + 8r'^2 - 16r'^3 + 32r'^4 + \cdots \quad (18)$$

$$\lambda^2 = 1 - 16r' + 128r'^2 - 704r'^3 + 3072r'^4 + \cdots$$

$$k_0 = 1 - 32r' + 256r'^2 - 1408r'^3 + 6144r'^4 + \cdots \quad (19)$$

$$\sqrt{k_0} = 1 - 16r' - 704r'^3 - 8192r'^4 + \cdots$$

$$\epsilon = 4r' + 32r'^2 + 432r'^3 + 6912r'^4 + \cdots \quad (20)$$

$$q' = 4r' + 32r'^2 + 432r'^3 + 6912r'^4 + \cdots \quad (21)$$

$$\ln(q)' = \ln(4) + \ln(r') + 8r' + 76r'^2 + 11034.7r'^3 + \cdots$$

$$C_0 = 4\left[\frac{2}{\delta} - 1 - \frac{\ln(4) + 8r' + 76r'^2 + 1034.7r'^3}{\pi} + \cdots\right]. \quad (22)$$

The first two terms of this expansion, except for a factor of 8

$$1/\delta - .5 - \ln(2)/\pi = \frac{(b+a)}{2(b-a)} - \frac{\ln(2)}{\pi}$$

is given in [1, p. 105].

The values of C_0 given by (22) are tabulated as $C(\delta)$ on the third row of Table I. It is seen from Table I that (15) and (22) give five place agreement with the values given by (7).

IV. THE COMPARISON

In a former paper, an expansion is given, in terms of δ, for the capacitance of the Bowman Square, where the geometry differs from the geometry in this short paper in that the inner square is rotated through 45°.[2] The first five terms of that expansion [4, (28)] are

$$C_0 = 8\pi/(4\log 1/\delta) + 1.02613 - .19114\delta^4 - .08188\delta^8 - .052318\delta^{12} + \cdots. \quad (23)$$

If ρ is replaced by $\delta/\sqrt{2}$ in (15), to account for the difference in the definitions of δ and ρ

$$C_0 = 8\pi/(4\log(1/\delta) + 1.02614 + .19114\delta^4 - .08186\delta^8 + .05264\delta^{12} + \cdots. \quad (24)$$

[2]For a discussion of the conformal mapping involved in these solutions, see [5, pp. 87, 88].

Is it possible that these expansions are identical except for the sign of the odd powers of δ'^{4}?

REFERENCES

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Continuous Spectrum and Characteristic Modes of the Slot Line in Free Space

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***Abstract*—The continuous spectrum of the slot in an infinite ground plane is expressed in terms of the Mathieu functions. The so-called characteristic values and field modes of the slot are stated explicitly.**

I. CONTINUOUS SPECTRUM REPRESENTATION

In a recent paper [1] a general eigenspectrum construction method for open waveguides was presented. As an illustration, the case of the slotted screen was considered. Extensive analysis of the same problem was also undertaken in [2] and subsequent publications by the same authors. The purpose of this note is to state in closed form the eigenmodes, the characteristic slot-field modes and values for the aforementioned geometry. These may be used to verify numerical solutions, as basis functions for more complex, nonseparable geometries, or to investigate slot-line discontinuities.

Consider an infinite slot of width a in a perfectly conducting, zero-thickness screen. Let the center line of the slot define the z-axis, and the x-axis lie in the plane of the screen. The fields in this structure can be represented in terms of a complete, orthonormal set of z-guided eigenfunctions, each satisfying appropriate boundary conditions on the screen. The transverse-to-z cross section of the structure is unbounded and homogeneously filled; the eigenspectrum is continuous and allows decomposition into TE_z and TM_z components [3]

For a complete description of the notation used here for the elliptic cylindrical coordinates and the Mathieu functions the reader is referred to [4].

A. TM_z Eigenmodes

The transverse electric field of a TM_z eigenmode can be represented as the gradient of a scalar function $\Phi_m(h, \cosh\mu, \cos\theta)$, where $h = \frac{1}{2}k_t a$, $0 \leq k_t \leq \infty$ is the continuous spectral variable, m is the discrete index associated with the angular solutions and μ, θ denote, respectively, the radial and angular elliptic coordinates.

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Following the approach employed in [5], [6] for the case of a ridged elliptical waveguide, eigenmodes with even or odd symmetry with respect to the plane of the screen are distinguished.

The even solutions are given by

$$\begin{aligned}\Phi^e_m(h, \cosh\mu, \cos\theta) &= Po_m(h, \cosh\mu) So_m(h, \cos\theta)\\ Po_m(h, \cosh\mu) &= No'_m(h, 1) Jo_m(h, \cosh\mu)\\ &\quad - Jo'_m(h, 1) No_m(h, \cosh\mu)\end{aligned}$$

and the odd solutions by

$$\begin{aligned}\Phi^o_m(h, \cosh\mu, \cos\theta) &= Qo_m(h, \cosh\mu) So_m(h, \cos\theta)\\ Qo_m(h, \cosh\mu) &= \frac{Jo_m(h, \cosh\mu)}{Jo'_m(h, 1)}\end{aligned}$$

where $So_m(h, \cos\theta)$ is the m-th odd angular Mathieu function, $Jo_m(h, \cosh\mu)$ and $No_m(h, \cosh\mu)$ are, respectively, the radial Mathieu functions of the first and second kind associated with the odd angular solution, and prime denotes differentiation with respect to μ. The boundary conditions on the screen are satisfied by virtue of the fact that $So_m(h, \cos\theta) = 0$ at $\theta = 0, \pi$. Among other properties of the two solutions are the following facts

$$\begin{aligned}&Po_m(h, 1) = 1 \quad Po'_m(h, 1) = 0\\ &Qo_m(h, 1) = 0 \quad Qo'_m(h, 1) = 1\\ &\mathcal{W}(Po_m(h, \cosh\mu), Qo_m(h, \cosh\mu)) = 1\end{aligned}$$

where $\mathcal{W}$ denotes the Wronskian.

B. TE_z Eigenmodes

The transverse magnetic field of a TE_z eigenmode can be represented as the gradient of a scalar function $\Psi_m(h, \cosh\mu, \cos\theta)$, where the previously introduced notation is applicable.

The even solutions are given by

$$\begin{aligned}\Psi^e_m(h, \cosh\mu, \cos\theta) &= Pe_m(h, \cosh\mu) Se_m(h, \cos\theta)\\ Pe_m(h, \cosh\mu) &= Je_m(h, 1) Ne_m(h, \cosh\mu)\\ &\quad - Ne_m(h, 1) Je_m(h, \cosh\mu)\end{aligned}$$

and the odd solutions by

$$\begin{aligned}\Psi^o_m(h, \cosh\mu, \cos\theta) &= Qe_m(h, \cosh\mu) Se_m(h, \cos\theta)\\ Qe_m(h, \cosh\mu) &= \frac{Je_m(h, \cosh\mu)}{Je_m(h, 1)}\end{aligned}$$

where $Se_m(h, \cos\theta)$ is the mth even angular Mathieu function, $Je_m(h, \cosh\mu)$ and $Ne_m(h, \cosh\mu)$ are, respectively, the radial Mathieu functions of the first and second kind associated with the even angular solution, and prime denotes differentiation with respect to μ. The boundary conditions on the screen are satisfied because $\frac{\partial Se_m(h, \cos\theta)}{\partial\theta} = 0$ at $\theta = 0, \pi$. Additional properties of the two solutions include

$$\begin{aligned}&Pe_m(h, 1) = 0 \quad Pe'_m(h, 1) = 1\\ &Qe_m(h, 1) = 1 \quad Qe'_m(h, 1) = 0\\ &\mathcal{W}(Pe_m(h, \cosh\mu), Qe_m(h, \cosh\mu)) = 1.\end{aligned}$$